∞ -Variate Integration

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Presentation based on papers co-authored with

A. Gilbert, M. Gnewuch, M. Hefter,
A. Hinrichs, P. Kritzer, F. Y. Kuo,
F. Pillichshammer, L. Plaskota,
K. Ritter, I. H. Sloan, H. Woźniakowski

INFORMATION-BASED COMPLEXITY approach very nicely explained in Houman Owhadi's tutorial: many thanks Houman

Briefly: instead of working with specific functions, **IBC** deals with problems on whole spaces of functions and tries to determine the **complexity**, i.e., the minimal cost, and (almost) **optimal algorithms**.

This is done in various settings including: worst case, average case and randomized settings

"Classical" Integration Problem

Given a normed space F of functions on $[0,1]^d$

approximate
$$I(f) = \int_{D^d} f(\boldsymbol{x}) \, \mathrm{d} \boldsymbol{x}$$

by cubatures
$$Q_n(f) = \sum_{j=1}^n f(oldsymbol{t}_{n,j}) \cdot a_{n,j}$$

with small error $\|I - Q_n\|$ and

(if possible) small cost

Classical Methods are Extremely Bad!!!! even for Finitely Many Variables.

(Product) Trapezoidal T_n with n samples has error

$$\operatorname{error}(T_n; d=2) \simeq \frac{1}{n} \quad d=2 \text{ variables}$$

$$error(T_n; d = 360) \simeq \frac{1}{n^{2/360}}$$
 $d = 360$ variables

E.g., for 360 variables, it needs

 $n \sim 20^{180}$ to get only 1 digit of accuracy

 $20^{180} =$

For Classical (Isotropic) Spaces

One Cannot Do Better

"Curse of Dimensionality"

For Classical (Isotropic) Spaces

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"Curse of Dimensionality"

To Break this Curse

Different Spaces Are Needed

Weighted Spaces treat different variables differently

Motivating Example

Compute expectation $\mathbb{E}(g(X(t_0)))$ for

stochastic process
$$\boldsymbol{X}(t) = \sum_{j=1}^{\infty} x_j \, \xi_j(t)$$

Equivalent to computing an integral of

$$f(x_1, x_2, \dots) = g\left(\sum_{j=1}^{\infty} x_j \,\xi_j(t_0)\right)$$

 $g\left(\sum_{j=1}^{\infty} x_j \,\xi_j(t_0)\right)$ In

"importance" of x_j

is quantized by the size of $|\xi_j(t_0)|$.

The larger $|\xi_j(t_0)|$ the more important x_j .

Although there are results for quite general spaces and problems

we present results for Integration over a special space ${\cal F}$

 ${\cal F}$ is the γ -weighted Banach space of functions with dominating mixed derivatives of order one bounded in L_p -norm

Notation:

 \mathfrak{w} finite subsets of \mathbb{N}_+

listing the "variables in action"

Given
$$\boldsymbol{x} = (x_1, x_2, \dots), \qquad \boldsymbol{x}_{\boldsymbol{w}} = (x_j : j \in \boldsymbol{w})$$

$$oldsymbol{x}_{\mathfrak{w}}; oldsymbol{0}] \,=\, (y_1, y_2, \dots) \hspace{1.5cm} ext{with} \hspace{1.5cm} y_j \,=\, \left\{ egin{array}{cc} x_j & ext{if} \ j \in \mathfrak{w}, \ 0 & ext{if} \ j
otin \mathfrak{w} \end{array}
ight.$$

$$f^{(\mathfrak{w})} = \frac{\partial^{|\mathfrak{w}|}}{\partial \boldsymbol{x}_{\mathfrak{w}}} f = \prod_{j \in \mathfrak{w}} \frac{\partial}{\partial x_j} f$$

Domain: $D^{\mathbb{N}}$ set of sequences $(x_j)_{j \in \mathbb{N}}$ with $x_j \in D$; for simplicity D = [0, 1].

> \mathcal{F} the Banach space of $f: D^{\mathbb{N}} \to \mathbb{R}$ endowed with the norm

$$\|f\|_{\mathcal{F}} = \left(\sum_{\mathfrak{w}} \gamma_{\mathfrak{w}}^{-p} \|f^{(\mathfrak{w})}([\cdot_{\mathfrak{w}}; \mathbf{0}])\|_{L_p(D^{|\mathfrak{w}|})}^p\right)^{1/p} < \infty$$

Here $p \in [1, \infty]$ and $\gamma_{\mathfrak{w}} \geq 0$ are *weights*

For simplicity

Product Weights introduced by

[Sloan and Woźniakowski 1998]

$$\gamma_{\mathfrak{w}} = c \prod_{j \in \mathfrak{w}} \gamma_j \qquad (\gamma_j = j^{-\beta})$$

For 'motivating example' we have

$$\gamma_{\mathfrak{w}} \simeq \prod_{j \in \mathfrak{w}} |\xi_j(t_0)|^{\alpha}$$

Integration Problem

APPROXIMATE:

$$\begin{aligned} \mathcal{I}(f) &:= \int_{D^{\mathbb{N}}} f(\boldsymbol{x}) \, \mathrm{d}^{\mathbb{N}} \boldsymbol{x} \\ &= \lim_{\boldsymbol{d} \to \infty} \int_{D^{\boldsymbol{d}}} f(x_1, \dots, x_{\boldsymbol{d}}, 0, 0, \dots) \, \mathrm{d}[x_1, \dots, x_{\boldsymbol{d}}] \end{aligned}$$

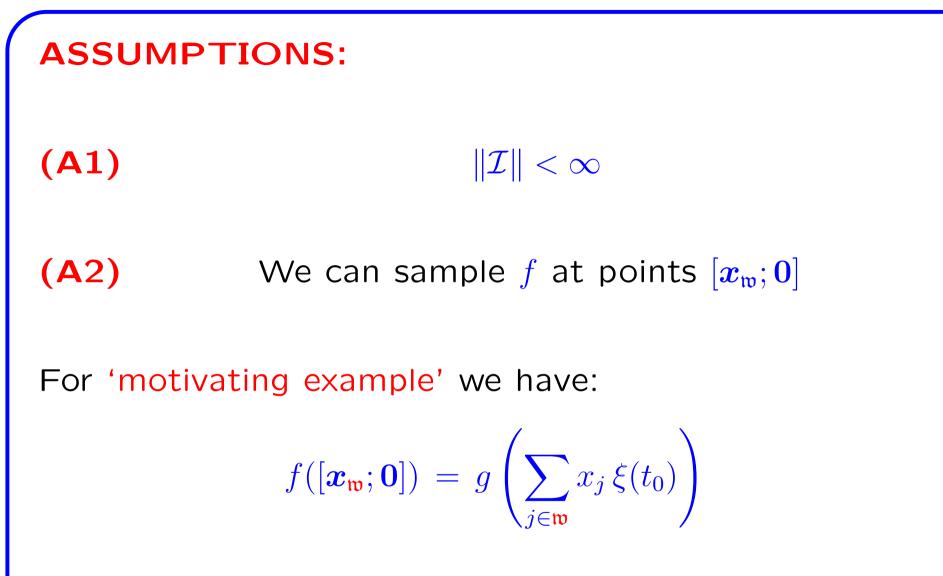
Integration Problem

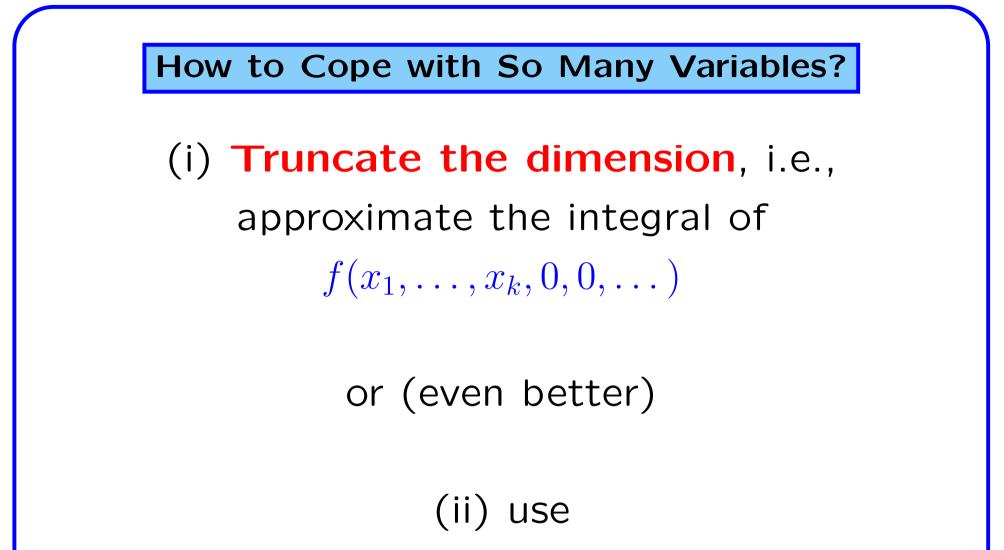
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We have

$$\|\mathcal{I}\| = \left(\sum_{\mathfrak{w}} \gamma_{\mathfrak{w}}^{p^*} / (1+p^*)^{|\mathfrak{w}|}\right)^{1/p^*} \quad \left(=\max_{\mathfrak{w}} \gamma_{\mathfrak{w}} \text{ for } p=1\right)$$





Multivariate Decomposition Method

Low Truncation Dimension

[Kritzer, Pillichshammer, W. 2016] \subset [Hinrichs, Kritzer, Pillichshammer, W.] Let

$$f_{k}(x_{1},\ldots,x_{k}) = f(x_{1},\ldots,x_{k},0,0,\ldots)$$

Given the error demand $\varepsilon > 0$, $\operatorname{dim}^{\operatorname{trnc}}(\varepsilon) \varepsilon$ -truncation dimension the smallest k such that $|\mathcal{I}(f) - \mathcal{I}(f_k)| \leq \varepsilon ||f||_{\mathcal{F}}$ for all $f \in \mathcal{F}$

Our concept of **Truncation Dimension** is different than the one in Statistics!!!

If

 $|\mathcal{I}(f) - \mathcal{I}(f_k)| \le \varepsilon ||f||_{\mathcal{F}}$ and $|\mathcal{I}(f_k) - Q_k(f_k)| \le \varepsilon ||f||_{\mathcal{F}}$

then

$$|\mathcal{I}(f) - Q_{\mathbf{k}}(f_{\mathbf{k}})| \leq 2\varepsilon ||f||_{\mathcal{F}}$$

Hence

the smaller $dim(\varepsilon)$ the better

Special Case:
$$\gamma_{\mathfrak{w}} = c \prod_{j \in \mathfrak{w}} j^{-\beta}$$

$$\operatorname{dim}^{\operatorname{trnc}}(\varepsilon) \leq \min\left\{\ell : \sum_{j=\ell+1} j^{-\beta p^*} \leq \frac{p^*+1}{c^{p^*}} \ln(1/(1-\varepsilon^{p^*}))\right\}$$
$$= O\left(\varepsilon^{-1/(\beta-1+1/p)}\right)$$

for p > 1 and

$$\operatorname{dim}^{\operatorname{trnc}}(\varepsilon) = \left\lceil \left(\frac{c}{\varepsilon}\right)^{1/\beta} \right\rceil - 1$$

for p = 1

| Specific Values of $\dim^{trnc}(\varepsilon)$ for $p=1$ and $\gamma_{\mathfrak{w}} = \prod_{j \in \mathfrak{w}} j^{-\beta}$ | | | | | | | | | |
|---|---|-----------|-----------|-----------|-----------|-----------|-------------|--|--|
| | arepsilon | 10^{-1} | 10^{-2} | 10^{-3} | 10^{-4} | 10^{-5} | | | |
| | $\mathbf{dim}^{\mathbf{trnc}}(\varepsilon)$ | 2 | 9 | 31 | 99 | 316 | $\beta = 2$ | | |
| | $\mathbf{dim}^{\mathbf{trnc}}(\varepsilon)$ | 2 | 4 | 9 | 21 | 46 | $\beta = 3$ | | |
| | $\mathbf{dim}^{\mathbf{trnc}}(\varepsilon)$ | 1 | 3 | 5 | 9 | 17 | $\beta = 4$ | | |

For instance, for the error demand $\varepsilon = 10^{-3}$ with $\beta = 4$, only five variables instead of ∞ -many! Worst Case Error of QMC or Sparse Grids Methods is:

$$\leq O\left(\frac{\ln^4 n}{n}\right)$$



MDM replaces one ∞ -variate integral bv only few integrals each with only few variables

Introduced in [Kuo, Sloan, W., and Woźniakowski 2010]

Any function *f* has the unique **anchored decomposition**

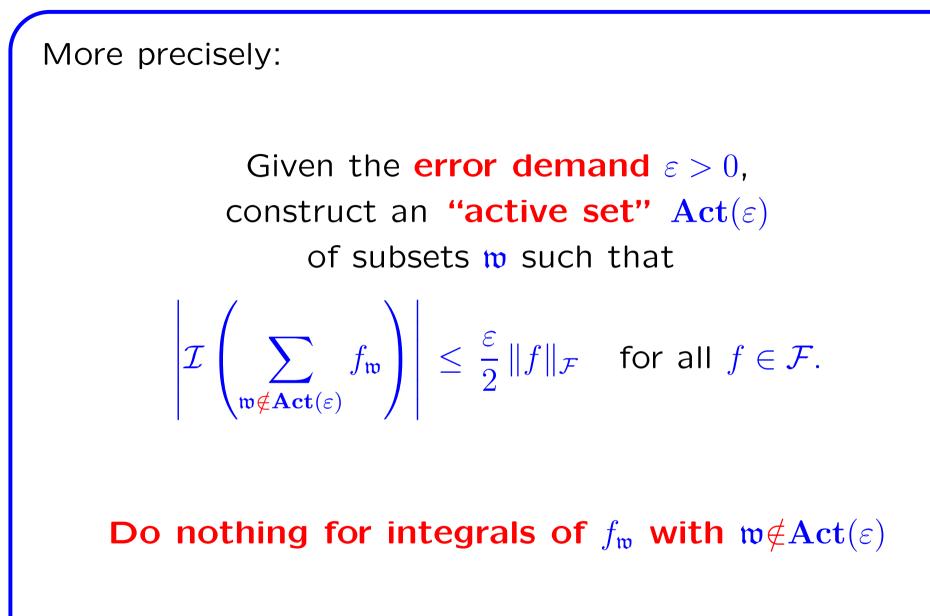
$$f(\boldsymbol{x}) = \sum_{\boldsymbol{w}} f_{\boldsymbol{w}}(\boldsymbol{x}_{\boldsymbol{w}}),$$

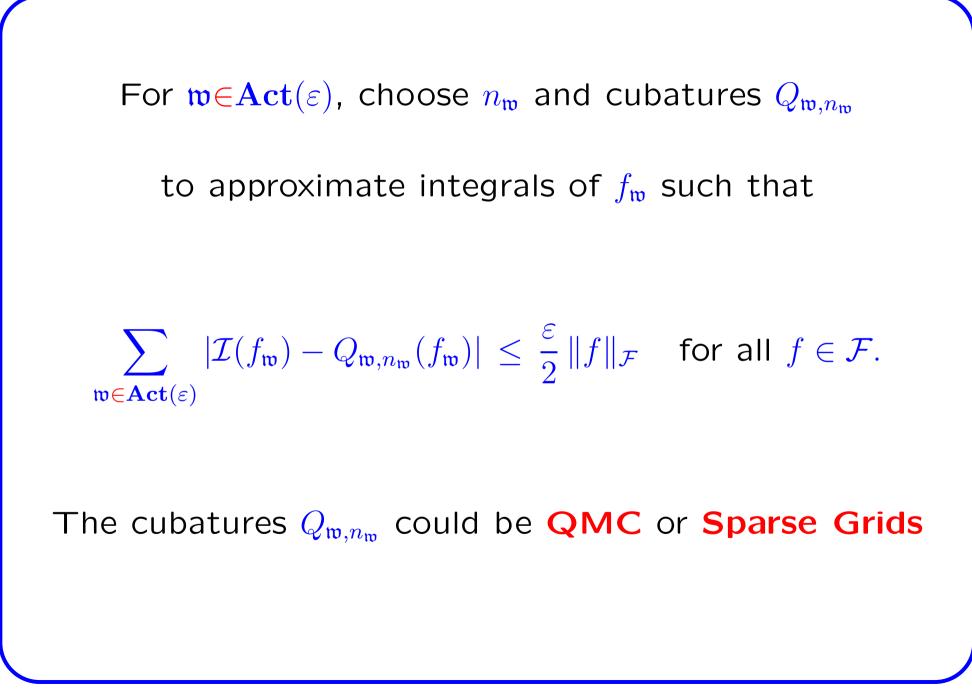
where f_{w} depends only on x_{j} with $j \in w$ and vanishes if $x_{j} = 0$.

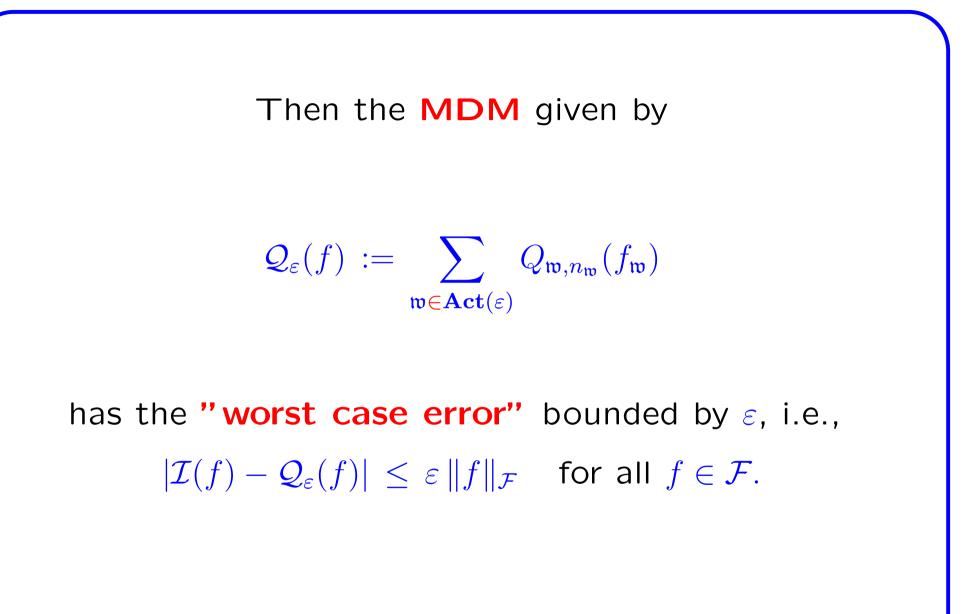
General Idea:

- Select the "most important" w's

- Approximate Integrals of $f_{\mathfrak{w}}$ only for the selected \mathfrak{w} 's







How about the **COST**?

The number of integrals to approximate is small:

$$\operatorname{\mathbf{card}}(\varepsilon) := |\operatorname{\mathbf{Act}}(\varepsilon)| = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

Each $f_{\mathfrak{w}}$ depends on only $|\mathfrak{w}|$ variables. The largest number of variables is also small:

 $\dim(\varepsilon) := \max \{ |\mathfrak{w}| : \mathfrak{w} \in \operatorname{Act}(\varepsilon) \} = \mathcal{O}(???)$

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$$\dim(\varepsilon) := \max\{|\mathfrak{w}| : \mathfrak{w} \in \operatorname{Act}(\varepsilon)\} = \mathcal{O}\left(\frac{\ln(1/\varepsilon)}{\ln(\ln(1/\varepsilon))}\right)$$

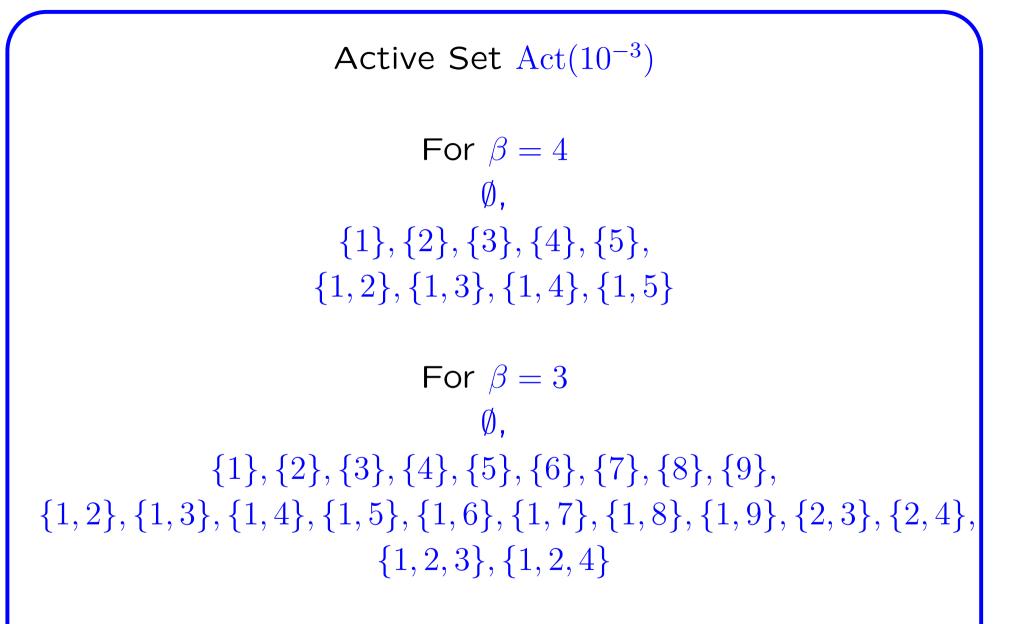
[Plaskota and W. 2011]

$dim(\varepsilon)$ is like Superposition Dimension in Statistics

Very efficient algorithm to construct $Act(\varepsilon)$ in [Gilbert and W. 2016] Specific Values of $\dim(\varepsilon)$ and $card(\varepsilon)$ for p = 1 and $\gamma_{\mathfrak{w}} = \prod_{i \in \mathfrak{w}} j^{-\beta}$

| arepsilon | 10^{-1} | | 10^{-2} | | 10^{-3} | | 10^{-4} | | 10^{-5} | | |
|-----------|-----------|---|-----------|---|-----------|----|-----------|----|-----------|----|-------------|
| | | | | | | | | | | | $\beta = 2$ |
| | | | | | | | | | | | $\beta = 3$ |
| | 1 | 2 | 2 | 6 | 2 | 10 | 3 | 26 | 3 | 50 | $\beta = 4$ |

For instance, for $\varepsilon = 10^{-3}$ with $\beta = 4$ it is sufficient to approximate **10 integrals with at most 2 variables!**



For $\beta = 2$

31 of integrals with 1 variable54 of integrals with 2 variables26 of integrals with 3 variables2 of integrals with 4 variables

$$\emptyset, \\ \{1\}, \dots, \{31\}, \\ \{1, 2\}, \dots, \{1, 31\}, \{2, 3\}, \dots, \{2, 15\}, \\ \{3, 4\}, \dots, \{3, 10\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{5, 6\}, \\ \{1, 2, 3\}, \dots, \{1, 2, 15\}, \{1, 3, 4\}, \dots, \{1, 3, 10\}, \\ \{1, 4, 5\}, \{1, 4, 6\}, \{1, 4, 7\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 3, 5\}, \\ \{1, 2, 3, 4\}, \{1, 2, 3, 5\}$$

REMARK:

We do **NOT** know $f_{\mathfrak{w}}$ terms. However, we can sample them. Indeed due to [Kuo, Sloan, W., and Woźniakowski 2010b]

$$f_{\boldsymbol{\mathfrak{w}}}(\boldsymbol{x}_{\boldsymbol{\mathfrak{w}}}) = \sum_{\boldsymbol{\mathfrak{v}} \subseteq \boldsymbol{\mathfrak{w}}} (-1)^{|\boldsymbol{\mathfrak{w}}| - |\boldsymbol{\mathfrak{v}}|} f([\boldsymbol{x}_{\boldsymbol{\mathfrak{v}}}; \mathbf{0}])$$

requires

 $2^{|w|}$ samples of f

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requires

$$2^{|w|}$$
 samples of f

but from [Plaskota and W. 2011]

$$2^{|\mathbf{w}|} = O\left(\varepsilon^{\frac{-1}{\ln(\ln(1/\varepsilon))}}\right)$$
 is small for modest ε .



$$f \in \mathcal{F}^{ extsf{AVOVA}}$$
 iff $f(oldsymbol{x}) = \sum f_{\mathfrak{w}, \mathrm{A}}(oldsymbol{x}_{\mathfrak{w}})$

with
$$\int_D f_{\mathfrak{w},A}(\boldsymbol{x}_{\mathfrak{w}}) dx_j = 0$$
 and
 $\|f\|_{\mathcal{F}^{ANOVA}} = \left(\sum_{\mathfrak{w}} \gamma_{\mathfrak{w}}^{-p} \|f_{\mathfrak{w},A}^{(\mathfrak{w})}\|_{L_p}^p\right)^{1/p} < \infty$

ANOVA decomposition terms $f_{w,A}$ cannot be sampled, i.e., low truncation dimension and MDM might not be applicable.

Even worse: the 'easiest' (constant) term is **NOT known**;

it is the integral we want:

$$f_{\emptyset,\mathbf{A}} = \mathcal{I}(f)$$

HOWEVER

If the spaces are **EQUIVALENT**, then

efficient algorithms for anchored spaces

are also efficient for ANOVA spaces

This motivated the study of



For product weights $\gamma_{\mathfrak{w}} = \prod_{j \in \mathfrak{w}} j^{-\beta}$

$$\mathcal{F} = \mathcal{F}^{ANOVA}$$
 as sets.

For the imbedding $\imath : \mathcal{F} \hookrightarrow \mathcal{F}^{\text{ANOVA}}$ we have

$$\|\boldsymbol{\imath}\| = \|\boldsymbol{\imath}^{-1}\| \le \prod_{j=1}^{\infty} (1+j^{-\beta})$$

EQUIVALENCE iff $\beta > 1$

Research direction initiated in [Hefter and Ritter 2014], Hilbert spaces setting p = 2 and product weights

[Hefter, Ritter and W. 2016] $p \in \{1, \infty\}$ and general weights,

[Hinrichs and Schneider 2016] $p \in (1,\infty)$,

[Gnewuch, Hefter, Hinrichs, Ritter, and W. 2016] more general spaces,

[Kritzer, Pillichshammer, and W. 2017] sharp lower bounds,

[Hinrichs, Kritzer, Pillichshammer, and W. 2017] most general

GENERALIZATIONS

More General Domain: Any interval D including $D = \mathbb{R}$

More General Distributions μ on D: e.g., Exponential, Gaussian

More General Integrals: $\int_{\mathbb{R}^N} f(\boldsymbol{x}) \mu^{\mathbb{N}}(\mathrm{d}\boldsymbol{x})$

Other Linear Solution Operators: S(f) = ???e.g., Function Approximation, ODE's, PDE's

> General Information about f: $L_1(f), L_2(f), \ldots, L_n(f), \qquad L_j \in \mathcal{F}^*$

Bayesian Approach:

Endowing \mathcal{F} with Gaussian probability measure PROB and studying average case errors:

 $\int_{\mathcal{F}} \|\mathcal{S}(f) - Alg(L_1(f), \dots, L_n(f))\|_{\mathcal{S}(\mathcal{F})}^p \operatorname{PROB}(\mathrm{d}f)$

Similar results in [W. 2014]

Comments to Houman Owhadi's 1st talk:

[Traub, W., and Woźniakowski 1988] has a number of chapters devoted to the average, randomized and probabilistic settings for infinitely dimensional Hilbert and Banach spaces. They are based on a number of earlier papers. Currently there are 100's of IBC such papers, see e.g. 3 Volumes monograph: [E. Novak and H. Woźniakowski 2008-10]

On page 16, the **IBC Probabilistic Setting** was attributed to H. Woźniakowski's paper. However, as acknowledged in that paper, the results were based on some of the results of my paper: Optimal algorithms for linear problems with Gaussian measures, *Rocky Mountains J. of Math.* 1986.

where IBC Probabilistic Setting

has been introduced for the first time.

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Comment to Ilias Bilionis' talk:
          Research that seem to be related:
   IBC approach to PDE's with random coefficients
         by Ch. Shwab and his collaborators,
         e.g., F.Y.Kuo, D. Nuyens, I. H. Sloan
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THANK YOU FOR THE ATTENTION